Statistical Analysis of Corpus Data with R

A short introduction to regression and linear models

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Outline

1. Regression
   - Simple linear regression
   - General linear regression
2. Linear statistical models
   - A statistical model of linear regression
   - Statistical inference
3. Generalised linear models

Linear regression

- Can random variable Y be predicted from r. v. X?
  - ☞ focus on linear relationship between variables

- Linear predictor:

$$Y \approx \beta_0 + \beta_1 \cdot X$$

  - β_0 = intercept of regression line
  - β_1 = slope of regression line

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- Least-squares regression minimizes prediction error

$$Q = \sum_{i=1}^{n} \left[y_i - (\beta_0 + \beta_1 x_i) \right]^2$$

for data points $(x_1, y_1), \ldots, (x_n, y_n)$

Simple linear regression

- Coefficients of least-squares line

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} x_i y_i - n\overline{x}_n\overline{y}_n}{\sum_{i=1}^{n} x_i^2 - n\overline{x}_n^2}$$

$$\hat{\beta}_0 = \overline{y}_n - \hat{\beta}_1\overline{x}_n$$

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- Mathematical derivation of regression coefficients
  - minimum of $Q(\beta_0, \beta_1)$ satisfies $\partial Q / \partial \beta_0 = \partial Q / \partial \beta_1 = 0$
  - leads to normal equations (system of 2 linear equations)

$$-2 \sum_{i=1}^{n} \left[y_i - (\beta_0 + \beta_1 x_i) \right] = 0 \quad \rightarrow \quad \beta_0 n + \beta_1 \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$$

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  - regression coefficients = unique solution $\hat{\beta}_0, \hat{\beta}_1$

The Pearson correlation coefficient

- Measuring the "goodness of fit" of the linear prediction
  - variation among observed values of Y = sum of squares S_y^2
  - closely related to (sample estimate for) variance of Y

$$S_y^2 = \sum_{i=1}^{n} (y_i - \overline{y}_n)^2$$

  - residual variation wrt. linear prediction: $S_{\text{resid}}^2 = Q$

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- Pearson correlation = amount of variation "explained" by X

$$R^2 = 1 - \frac{S_{\text{resid}}^2}{S_y^2} = 1 - \frac{\sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2}{\sum_{i=1}^{n} (y_i - \bar{y}_n)^2}$$

☞ correlation vs. slope of regression line

$$R^2 = \hat{\beta}_1(y \sim x) \cdot \hat{\beta}_1(x \sim y)$$

Multiple linear regression

- Linear regression with multiple predictor variables

$$Y \approx \beta_0 + \beta_1 X_1 + \cdots + \beta_k X_k$$

  minimises

$$Q = \sum_{i=1}^{n} \bigl[y_i - (\beta_0 + \beta_1 x_{i1} + \cdots + \beta_k x_{ik})\bigr]^2$$

  for data points $\bigl(x_{11}, \ldots, x_{1k}, y_1\bigr),\ \ldots,\ \bigl(x_{n1}, \ldots, x_{nk}, y_n\bigr)$

- Multiple linear regression fits *n*-dimensional hyperplane instead of regression line

Multiple linear regression: The design matrix

- Matrix notation of linear regression problem

$$\mathbf{y} \approx \mathbf{Z}\beta$$

- "Design matrix" **Z** of the regression data

$$\mathbf{Z} = \begin{bmatrix} 1 & x_{11} & x_{12} & \cdots & x_{1k} \\ 1 & x_{21} & x_{22} & \cdots & x_{2k} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{n1} & x_{n2} & \cdots & x_{nk} \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} y_1 & y_2 & \ldots & y_n \end{bmatrix}'$$

$$\beta = \begin{bmatrix} \beta_0 & \beta_1 & \beta_2 & \ldots & \beta_k \end{bmatrix}'$$

☞ $\mathbf{A}'$ denotes transpose of a matrix; $\mathbf{y}, \beta$ are column vectors

General linear regression

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- Residual error

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- Leads to regression coefficients

$$\hat{\beta} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{y}$$

General linear regression

- Predictor variables can also be functions of the observed variables ➜ regression only has to be linear in coefficients β

- E.g. polynomial regression with design matrix

$$\mathbf{Z} = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^k \\ 1 & x_2 & x_2^2 & \cdots & x_2^k \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^k \end{bmatrix}$$

corresponding to regression model

$$Y \approx \beta_0 + \beta_1 X + \beta_2 X^2 + \cdots + \beta_k X^k$$

Linear statistical models

- Linear statistical model (ϵ = random error)

$$Y = \beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k + \epsilon$$
$$\epsilon \sim N(0, \sigma^2)$$

  - ☞ $x_1, \ldots, x_k$ are not treated as random variables!
  - $\sim$ = "is distributed as"; $N(\mu, \sigma^2)$ = normal distribution

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- Assumptions
  - error terms ϵ_i are i.i.d. (independent, same distribution)
  - error terms follow normal (Gaussian) distributions
  - equal (but unknown) variance σ^2 = homoscedasticity

Linear statistical models

- Probability density function for simple linear model

$$\Pr(\mathbf{y} \mid \mathbf{x}) = \frac{1}{(2\pi\sigma^2)^{n/2}} \cdot \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2 \right]$$

  - $\mathbf{y} = (y_1, \ldots, y_n)$ = observed values of Y (sample size n)
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- Log-likelihood has a familiar form:

$$\log \Pr(\mathbf{y} \mid \mathbf{x}) = C - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2 \propto Q$$

➥ MLE parameter estimates $\hat{\beta}_0, \hat{\beta}_1$ from linear regression

Statistical inference for linear models

- Model comparison with ANOVA techniques
  - Is variance reduced significantly by taking a specific explanatory factor into account?
  - intuitive: proportion of variance explained (like R^2)
  - mathematical: F statistic ➜ p-value

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  - t-tests ($H_0 : \beta_j = 0$) and confidence intervals for β_j
  - confidence intervals for new predictions

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- Categorical factors: dummy-coding with binary variables
  - e.g. factor x with levels *low*, *med*, *high* is represented by three binary dummy variables $x_{\text{low}}, x_{\text{med}}, x_{\text{high}}$
  - one parameter for each factor level: $\beta_{\text{low}}, \beta_{\text{med}}, \beta_{\text{high}}$
  - NB: β_{low} is "absorbed" into intercept β_0
    model parameters are usually $\beta_{\text{med}} - \beta_{\text{low}}$ and $\beta_{\text{high}} - \beta_{\text{low}}$

  ☞ mathematical basis for standard ANOVA

Interaction terms

- Standard linear models assume independent, additive contribution from each predictor variable x_j ($j = 1, \ldots, k$)
- Joint effects of variables can be modelled by adding interaction terms to the design matrix (+ parameters)
- Interaction of numerical variables (interval scale)
  - interaction term for variables x_i and x_j = product $x_i \cdot x_j$
  - e.g. in multivariate polynomial regression: $Y = p(x_1, \ldots, x_k) + \epsilon$ with polynomial p over k variables
- Interaction of categorical factor variables (nominal scale)
  - interaction of x_i and x_j coded by one dummy variable for each combination of a level of x_i with a level of x_j
  - alternative codings e.g. to have separate parameters for independent additive effects of x_i and x_j
- Interaction of categorical factor with numerical variable

Generalised linear models

- Linear models are flexible analysis tool, but they ...
  - only work for a numerical response variable (interval scale)
  - assume independent (i.i.d.) Gaussian error terms
  - assume equal variance of errors (homoscedasticity)
  - cannot limit the range of predicted values
- Linguistic frequency data problematic in all four respects
  - ☞ each data point y_i = frequency f_i in one text sample
  - f_i are discrete variables with binomial distribution (or more complex distribution if there are non-randomness effects)
  - ☞ linear model uses relative frequencies $p_i = f_i / n_i$
  - Gaussian approximation not valid for small text size n_i
  - sampling variance depends on text size n_i and "success probability" π_i (= relative frequency predicted by model)
  - model predictions must be restricted to range $0 \leq p_i \leq 1$

➥ General*ised* linear models (GLM)

Generalised linear model for corpus frequency data

- Sampling family (binomial)

$$f_i \sim B(n_i, \pi_i)$$

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➥ Estimation and ANOVA based on likelihood ratios

☞ iterative methods needed for parameter estimation